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Factorization of Disconjugate Higher-Order Sturm-Liouville Difference Operators

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Abstract—Using a recently proved equivalence between disconjugacy of the $2n^{\text{th}}$-order difference equation

$$L(y)_{k+n} := \sum_{\nu=0}^{n}(-1)^\nu \Delta^\nu \left(r_k^{(\nu)} \Delta^\nu y_{k+n-\nu}\right) = 0,$$

and solvability of the corresponding Riccati matrix difference equation, it is shown that the equation $L(y) = 0$ is disconjugate on a given interval if and only if the operator L admits the factorization of the form

$$L(y)_{k+n} = M^*(c_k M(y)_k)_{k+n},$$

where M and its adjoint M^* are certain n^{th}-order difference operators and c_k is a sequence of positive numbers. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Sturm-Liouville operator, Riccati equation, Linear Hamiltonian difference system, Disconjugacy.

1. INTRODUCTION

In this paper, we deal with factorization of the self-adjoint Sturm-Liouville difference operators

$$L(y)_{k+n} = \sum_{\nu=0}^{n}(-1)^\nu \Delta^\nu \left(r_k^{(\nu)} \Delta^\nu y_{k+n-\nu}\right), \tag{1.1}$$

where $r_k^{(\nu)}$, $\nu = 0, 1, \dots, n$, are real-valued sequences, $r_k^{(n)} \neq 0$, $k \in \{0, \dots, N\} =: J$, $N > n$.

Concerning the continuous counterpart of (1.1),

$$l(y) = \sum_{\nu=0}^{n}(-1)^\nu \left(r_\nu(t) y^{(\nu)}\right)^{(\nu)}, \tag{1.2}$$

where $t \in I := [a, b] \subset \mathbf{R}$, $r_n(t) > 0$ on I, it is known that the equation $l(y) = 0$ is disconjugate on I (i.e., no nontrivial solution of (1.2) has more than two different zeros of multiplicity n in I) if and only if the operator l admits the factorization of the form

$$l(y) = m^*(r(t)m(y)), \tag{1.3}$$

where

$$m(y) = y^{(n)} + \sum_{j=0}^{n-1} q_j(t) y^{(j)}, \qquad m^*(z) = (-1)^n z^{(n)} + \sum_{j=0}^{n-1}(-1)^j (q_j(t) z)^{(j)}$$

are certain (mutually adjoint) n^{th}-order differential operators, see [1].

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Here, we establish a similar statement for disconjugate difference operators (1.1). The problem of the definition of disconjugacy for the difference equation

$$L(y) = 0, \tag{1.4}$$

or more generally, for the linear Hamiltonian difference systems

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A_k^T u_k, \tag{1.5}$$

in such a way that disconjugacy implies nonnegativity of the associated discrete quadratic functionals

$$\mathcal{J}(y) = \sum_{k=0}^{N} \left\{ \sum_{\nu=0}^{n} r_k^{(\nu)} (\Delta^\nu y_{k+n-\nu})^2 \right\}, \tag{1.6}$$

$$\mathcal{F}(x,u) = \sum_{k=0}^{N} \left\{ u_k^T B_k u_k + x_{k+1}^T C_k x_{k+1} \right\} \tag{1.7}$$

remained open for a relatively long time and only some partial results were known, see [2] and the reference given therein. Recently, Bohner [3] proved the discrete version of the so-called Reid Roundabout Theorem for linear Hamiltonian difference systems (1.5) (further LHdS), this theorem states, roughly speaking, that disconjugacy of (1.5) is equivalent to nonnegativity of $\mathcal{F}$ in the class of the sequence (x_k, u_k) satisfying

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad x_0 = 0 = x_{N+1},$$

and this is equivalent to the existence of a symmetric solution of the discrete Riccati matrix equation

$$Q_{k+1} = C_k + \left(I - A_k^T\right) Q_k (I + B_k Q_k)^{-1} (I - A_k). \tag{1.8}$$

This general theorem, when applied to (1.6), says that the functional $\mathcal{J}$ is nonnegative in the class of sequences y_k satisfying $y_0 = \cdots = y_{n-1} = 0 = y_{N+1} = \cdots = y_{N+n}$ if and only if no solution of (1.4) has in J two or more generalized zeros of multiplicity n (in the sense of Hartman's definition [4], for more details, see the next section).

In this paper, we give another condition which is equivalent to disconjugacy of (1.4), namely, the existence of an n^{th}-order linear difference operator

$$M(y) := \Delta^n y_k - a_k^{(n-1)} \Delta^{n-1} y_{k+1} - \cdots - a_k^{(1)} \Delta y_{k+n-1} - a_k^{(0)} y_{k+n}, \tag{1.9}$$

with $1 - \sum_{\nu=0}^{n-1} a_k^{(\nu)} \neq 0$ in J, and $c_k > 0$ such that the operator L admits the factorization of the form

$$L(y)_{k+n} = M^*(c_k M(y)_k)_{k+n},$$

where M^* is the adjoint operator of M.

The paper is organized as follows. In the next section, we recall basic properties of solutions of LHdS (1.5) and their relationship to (1.4) and (1.8). In Section 3, we present the main result of the paper and the last section is devoted to remarks and comments concerning related topics.

2. PRELIMINARIES

Throughout the paper, we use the following notation. If D is a symmetric matrix, the inequality $D \geq 0$ ($D > 0$) means that D is nonnegative (positive) definite. For any matrix V, by $V^\dagger$, we denote its Moore-Penrose pseudoinverse, i.e., the matrix such that both matrices

$VV^\dagger, V^\dagger V$ are symmetric and $VV^\dagger V = V$, $V^\dagger V V^\dagger = V^\dagger$. The symbol Δ denotes the usual forward difference operator $\Delta y_k = y_{k+1} - y_k$ and differences of higher-order are defined recurrently $\Delta^j y_k = \Delta(\Delta^{j-1})y_k$.

We start with basic properties of (1.5). Let $A, B, C : J \to \mathbf{R}^{n\times n}$ be real valued matrices such that B, C are symmetric, i.e., $B = B^T$, $C = C^T$, and the matrix $I - A$ is nonsingular (I denotes the identity matrix), $\tilde{A} := (I - A)^{-1}$. Simultaneously with (1.5), we consider its matrix version

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A_k^T U_k, \tag{2.1}$$

where X, U are $n \times n$ matrices. Clearly, if (X, U) is a solution of (2.1) and $c \in \mathbf{R}^n$, then $(x, u) = (Xc, Uc)$ solves (1.5).

Let $(X, U), (\tilde{X}, \tilde{U})$ be solutions of (2.1). Then, $X_k^T \tilde{U}_k - U_k^T \tilde{X}_k \equiv D$, where D is a constant $n\times n$ matrix. A solution (X, U) is said to be *self-conjoined* if $X^T U - U^T X \equiv 0$. System (1.5) is said to be *disconjugate* on J if the solution (X, U) of (2.1) given by the initial condition $(X_0, U_0) = (0, I)$ satisfies the conditions

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k, \tag{2.2.1}$$
$$D_k = X_k X_{k+1}^\dagger \tilde{A}_k B_k \geq 0, \tag{2.2.2}$$

for $k \in J$, where Ker denotes the kernel of the matrix indicated. Note, that if (2.2.1) holds then the matrices D_k are really symmetric as is shown in [3, Lemma 4]. Disconjugacy of (1.5) may be equivalently defined using the concept of the generalized zero point. A nontrivial solution (x, u) of (1.5) has *the generalized zero point* in the interval $(m, m+1]$, $m \in J$, if $x_m \neq 0$, there exists $c \in \mathbf{R}^n$ such that $x_{m+1} = \tilde{A}_m B_m c$ and $x_m^T c \leq 0$. System (1.5) is disconjugate on J if any nontrivial solution (x, u) satisfying $x_0 = 0$ has no generalized zero in J and any other solution has at most one generalized zero in J, see [5].

The crucial role in the proof of our factorization theorem plays the following statement proved in [6].

LEMMA 1. *System (1.5) is disconjugate on J if and only if there exist symmetric matrices Q : $J^* := [0, N+1] \to \mathbf{R}^{n\times n}$ such that $(I + BQ)$ is nonsingular $(I + BQ)^{-1}B \geq 0$ in J and Q satisfies the Riccati equation* (1.8).

Recall, that if (X, U) is a self-conjoined solution of (2.1) such that X is nonsingular, then $Q = UX^{-1}$ satisfies (1.8).

The higher-order Sturm-Liouville equation (1.4) is related to (1.5) by the following substitution. Let y_k be a solution of (1.4). Set

$$\begin{gathered}
x_k = \begin{pmatrix} x_k^{(1)} \\ x_k^{(2)} \\ \vdots \\ x_k^{(n-1)} \\ x_k^{(n)} \end{pmatrix} = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \\
u_k = \begin{pmatrix} u_k^{(1)} \\ u_k^{(2)} \\ \vdots \\ u_k^{(n-1)} \\ u_k^{(n)} \end{pmatrix} = \begin{pmatrix} r_k^{(1)} \Delta y_{k+n-1} - \Delta u_k^{(2)} \\ r_k^{(2)} \Delta^2 y_{k+n-2} - \Delta u_k^{(3)} \\ \vdots \\ r_k^{(n-1)} \Delta^{(n-1)} y_{k+1} - \Delta u_k^{(n)} \\ r_k^{(n)} \Delta^n y_k \end{pmatrix}.
\end{gathered} \tag{2.3}$$

Then, the pair of n-vectors (x,u) solves system (1.5) with

$$
\begin{aligned}
B_k &= \operatorname{diag}\left\{0,\dots,0,\frac{1}{r_k^{(n)}}\right\},\\
C_k &= \operatorname{diag}\left\{r_k^{(0)},\dots,r_k^{(n-1)}\right\}, \qquad (2.4)\\
A_k &= (A_k)_{i,j} = \begin{cases} 1, & \text{if } j=i+1,\ i=1,\dots,n-1,\\ 0, & \text{elsewhere.}\end{cases}
\end{aligned}
$$

We will also need the following transformation of LHdS which may be found in [7].

LEMMA 2. *Suppose that $H,K : J^* \to \mathbf{R}^{n\times n}$ are such that H and $H+BK$ are nonsingular and $H^TK = K^TH$ in J^*. Set*

$$x_k = H_k\tilde{x}_k, \qquad u_k = K_k\tilde{x}_k + H_k^{T-1}\tilde{u}_k. \tag{2.5}$$

Then, if (x,u) satisfy the first equation in (1.5), the new variables $(\tilde{x},\tilde{u})$ verify identities

$$\Delta\tilde{x}_k - \tilde{A}_k\tilde{x}_{k+1} - \tilde{B}_k\tilde{u}_k = 0, \tag{2.6.1}$$
$$H_{k+1}^T\left[\Delta u_k - C_kx_{k+1} + A_k^Tu_k\right] = \left[\Delta\tilde{u}_k - \tilde{C}_k\tilde{x}_{k+1} + \tilde{A}_k^T\tilde{u}_k\right], \tag{2.6.2}$$

where

$$
\begin{aligned}
\tilde{A}_k &= I - (H_k + B_kK_k)^{-1}(I - A_k)H_{k+1},\\
\tilde{B}_k &= (H_k + B_kK_k)^{-1}B_kH_k^{T-1}, \qquad (2.7)\\
\tilde{C}_k &= H_{k+1}^T\left[-K_{k+1} + C_kH_{k+1} + \left(I - A_k^T\right)K_k(H_k + B_kK_k)^{-1}(I - A_k)H_{k+1}\right].
\end{aligned}
$$

Finally, recall briefly the concept of the adjoint difference equation and system. For a more detailed treatment of this problem, see, e.g., [8]. Consider the first-order difference system

$$w_{k+1} = \mathcal{A}_kw_k, \tag{2.8}$$

where $w \in \mathbf{R}^n$ and $\mathcal{A} \in \mathbf{R}^{n\times n}$ are nonsingular. The system

$$z_k = \mathcal{A}_k^Tz_{k+1} \tag{2.9}$$

is called *the adjoint system* of (2.8) and a nonsingular $n\times n$ matrix W is the fundamental matrix of (2.8) if and only if $Z =: W^{T-1}$ is the fundamental matrix of (2.9). The substitution $w_k = (y_k, y_{k+1},\dots,y_{k+n-1})^T$ converts the n^{th}-order linear difference equation $M(y)=0$, with M given by (1.9), into the first-order system (2.8) and the assumption $1-\sum_{\nu=0}^{n-1}a_k^{(\nu)} \neq 0$ guarantees nonsingularity of the matrices $\mathcal{A}_k$ in this system. If $z_k = (z_k^{(1)},\dots,z_k^{(n)})^T$ is a solution of the adjoint system (2.9), one may directly verify that $y_k := z_k^{(n)}$ verifies the equation $M^*(y)=0$, hence, this equation is called *the (formally) adjoint equation* of $M(y)=0$ and M^* *the adjoint operator* of M.

3. MAIN RESULT

THEOREM 1. *The self-adjoint difference operator L given by (1.1) with $r_k^{(n)} \neq 0$ is disconjugate on J if and only if there exist an n^{th}-order linear difference operator M of the form (1.9) with $1-\sum_{\nu=0}^{n-1}a_k^{(\nu)} \neq 0$ and a positive sequence c_k such that*

$$L(y)_{k+n} = M^*(c_kM(y)_k)_{k+n}, \tag{3.1}$$

where

$$M^*(z)_{k+n} = (-1)^n \Delta^n z_k - \sum_{\nu=0}^{n-1} (-1)^\nu \Delta^\nu \left(a_k^{(\nu)} z_k\right)$$

is the adjoint operator of M.

PROOF. (i) *Necessity*. According to the equivalence between disconjugacy of (1.4) and nonnegativity of $\mathcal{J}$, it is sufficient to show that $\mathcal{J}(y) > 0$ for any nontrivial y satisfying

$$y_0 = \cdots = y_{n-1} = 0 = y_{N+1} = \cdots = y_{N+n}. \tag{3.2}$$

Observe, that if n-vectors x, u and the matrices A, B, C are given by (2.3),(2.4), then

$$\begin{aligned} L(y)_{k+n} &= e_1^T \left(-\Delta u_k + C_k x_{k+1} - A_k^T u_k\right), \\ e_\nu^T \left(-\Delta u_k + C_k x_{k+1} - A_k^T u_k\right) &= 0, \qquad \nu = 2, \dots, n, \end{aligned} \tag{3.3}$$

where $e_\nu \in \mathbf{R}^n$, $\nu = 1, \dots, n$, is the canonical basis of $\mathbf{R}^n$. Consequently, using this fact, summation by parts, and (3.3), we have for any y satisfying (3.2),

$$\begin{aligned} \sum_{k=0}^{N} y_{k+n} L(y)_{k+n} &= \sum_{k=0}^{N} x_{k+1}^T \left(-\Delta u_k + C_k x_{k+1} - A_k^T u_k\right) = x_k^T u_k \Big|_{k=0}^{N+1} \\ &\quad + \sum_{k=0}^{N} \left\{\Delta x_k^T u_k + x_{k+1}^T C_k x_{k+1} - x_{k+1}^T A_k^T u_k\right\} \\ &= \sum_{k=0}^{N} \left\{x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\right\} = \mathcal{F}(x,u) = \mathcal{J}(y). \end{aligned}$$

In the next computation, for notational convenience, we set $a_k^{(n)} := 1$. Then, again using summation by parts, for any $\nu \in \{0, \dots, n\}$,

$$\begin{aligned} y_{k+n} \Delta^\nu \left[a_k^{(\nu)} c_k M(y)_k\right] &= \sum_{j=1}^{\nu} \Delta \left\{\Delta^{j-1} y_{k+n-j} \Delta^{n-j} \left[a_k^{(\nu)} c_k M(y)_k\right]\right\} \\ &\quad + (-1)^\nu \Delta^\nu y_{k+n-\nu} a_k^{(\nu)} c_k M(y)_k. \end{aligned}$$

Consequently, for any y satisfying (3.2),

$$\begin{aligned} \sum_{k=0}^{N} y_{k+n} L(y)_{k+n} &= \sum_{k=0}^{n} y_{k+n} M^*(c_k M(y)_k)_{k+n} \\ &= \sum_{k=0}^{N} y_{k+n} \sum_{\nu=0}^{n} (-1)^\nu \Delta^\nu \left[a_k^{(\nu)} c_k M(y)_k\right] \\ &= \sum_{\nu=0}^{n} \sum_{j=1}^{\nu} (-1)^{\nu+j} \left[\Delta^{j-1} y_{k+n-j} \Delta^{n-j} (a_k c_k M(y)_k)\right]_{k=0}^{N+1} \\ &\quad + \sum_{k=0}^{N} \sum_{\nu=0}^{n} \Delta^\nu y_{k+n-\nu} a_k^{(\nu)} c_k M(y)_k = \sum_{k=0}^{N} c_k \left[M(y)_k\right]^2, \end{aligned}$$

and hence,

$$\mathcal{J}(y) = \sum_{k=0}^{N} c_k [M(y)_k]^2 \geq 0. \tag{3.4}$$

The equality $\mathcal{J}(y) = 0$ may happen only for $y \equiv 0$, since the equality $M(y)_k = 0$, $k \in J$, together with (3.2), implies $y \equiv 0$.

(ii) *Sufficiency*. Suppose that (1.4) is disconjugate on J. Then, the corresponding LHdS is disconjugate too, and by Lemma 1, there exists a symmetric solution Q of (1.8) such that $(I + B_k Q_k)$ is nonsingular and $(I + B_k Q_k)^{-1} B_k \geq 0$ for $k \in J$. Let $H = I$, $K = Q$ in (2.5), i.e.,

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} I & 0 \\ Q & I \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix},$$

then in (2.7),

$$\begin{aligned} \tilde{A}_k &= I - (I + B_k Q_k)^{-1}(I - A), \\ \tilde{B}_k &= (I + B_k Q_k)^{-1} B_k, \\ \tilde{C}_k &= -Q_{k+1} + C_k + \left(I - A_k^T\right) Q_k (I + B_k Q_k)^{-1}(I - A_k) = 0, \end{aligned}$$

where A, B, C are given by (2.4). Hence, we have

$$L(y)_{k+n} = e_1^T \left(-\Delta u_k + C_k x_{k+1} - A_k^T u_k\right) = e_1^T(-\Delta \tilde{u}_k - \tilde{A}_k^T \tilde{u}_k).$$

By a direct computation

$$(I + B_k Q_k)^{-1} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \dots & 1 & 0 \\ -\dfrac{Q_k^{(n,1)}}{Q_k^{(n,n)} + r_k^{(n)}} & -\dfrac{Q_k^{(n,2)}}{Q_k^{(n,n)} + r_k^{(n)}} & \dots & -\dfrac{Q_k^{(n,n-1)}}{Q_k^{(n,n)} + r_k^{(n)}} & \dfrac{r_k^{(n)}}{Q_k^{(n,n)} + r_k^{(n)}} \end{pmatrix},$$

$Q^{(i,j)}$, $i, j = 1, \dots, n$ being the entries of Q. Using the equality

$$\tilde{B}_k = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \dfrac{1}{Q_k^{(n,n)} + r_k^{(n)}} \end{pmatrix},$$

$$\tilde{A}_k = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ a_k^{(0)} & a_k^{(1)} & a_k^{(2)} & \dots & a_k^{(n-2)} & a_k^{(n-1)} \end{pmatrix},$$

where

$$a_k^{(0)} = \frac{Q_k^{(n,1)}}{Q_k^{(n,n)} + r_k^{(n)}}, \qquad a_k^{(j)} = \frac{Q_k^{(n,j+1)} - Q_k^{(n,j)}}{Q_k^{(n,n)} + r_k^{(n)}}, \qquad j = 1, \dots, n-1.$$

Nonnegative definiteness of $\tilde{B}$ together with nonsingularity of $(I + BQ)$ and $r^{(n)} \neq 0$ imply that $Q^{(n,n)} + r^{(n)} > 0$. Now, substituting this into (2.6.1) and taking into account that $\tilde{x} = x$ in (2.5), we get

$$\begin{aligned} \tilde{u}_k^{(n)} &= \left[Q_k^{(n,n)} + r_k^{(n)}\right] \left[\sum_{\nu=0}^{n-1} a_k^{(\nu)} \Delta^\nu y_{k+n-\nu} + \Delta^n y_k\right] \\ &= \left[Q_k^{(n,n)} + r_k^{(n)}\right] M(y)_k. \end{aligned} \tag{3.5}$$

Then,

$$1-\sum_{\nu=0}^{n-1} a_k^{(\nu)} = 1-\frac{Q_k^{(n,n)}}{Q_k^{(n,n)}+r_k^{(n)}} = \frac{r_k^{(n)}}{Q_k^{(n,n)}+r_k^{(n)}} \neq 0.$$

Concerning (2.6.2), we have

$$-\Delta\tilde{u}_k - \bar{A}_k\tilde{u}_k = -\Delta\begin{pmatrix} \tilde{u}_k^{(1)} \\ \tilde{u}_k^{(2)} \\ \vdots \\ \tilde{u}_k^{(n-1)} \\ \tilde{u}_k^{(n)} \end{pmatrix} - \begin{pmatrix} 0 & \dots & 0 & a_k^{(0)} \\ 1 & \dots & 0 & a_k^{(1)} \\ & \vdots & & \\ 0 & \dots & 0 & a_k^{(n-2)} \\ 0 & \dots & 1 & a_k^{(n-1)} \end{pmatrix}\begin{pmatrix} \tilde{u}_k^{(1)} \\ \tilde{u}_k^{(2)} \\ \vdots \\ \tilde{u}_k^{(n-1)} \\ \tilde{u}_k^{(n)} \end{pmatrix} = \begin{pmatrix} L(y)_{k+n} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

and from this equation, substituting successively for $u_k^{(1)},\dots,u_k^{(n-1)}$, we get

$$L(y)_{k+n} = (-1)^n\Delta^n\tilde{u}_k^{(n)} + \sum_{\nu=0}^{n-1}(-1)^\nu\Delta^\nu\left(a_k^{(\nu)}\tilde{u}_k^{(n)}\right) = M^*\left(\tilde{u}^{(n)}\right)_{k+n}.$$

The last equation together with (3.5) gives (3.1), where $c_k = Q_k^{(n,n)} + r_k^{(n)}$. ∎

4. CONCLUDING REMARKS

(i) The formula (3.1) may be seen from the point of view of the matrix theory as follows. Expanding the forward differences in (1.1), we have

$$L(y)_{k+n} = \sum_{\nu=0}^{n}\sum_{j=0}^{\nu}\binom{\nu}{j}\sum_{i=0}^{\nu}(-1)^{\nu+i+j}\binom{\nu}{i} r_{k+i}^{(\nu)} y_{k+n-\nu+i+j} =: \sum_{l=0}^{2n}\beta_{k+n}^{(l)} y_{k+l}.$$

The sequences $\beta_{k+n}^{(l)}$ may be computed explicitly, but we will not need explicit formulae for them. It is important that $\beta_{k+n}^{(n+l)} = \beta_{k+n+l}^{(n-l)}$, see [9, Section 14]. Hence, for any $\{y_k\}_{k=0}^{N+n}$ satisfying (3.2), we have

$$\begin{pmatrix} L(y)_n \\ \vdots \\ L(y)_N \end{pmatrix} = \mathcal{L}\begin{pmatrix} y_n \\ \vdots \\ y_N \end{pmatrix}, \qquad \mathcal{J}(y) = \begin{pmatrix} y_n \\ \vdots \\ y_N \end{pmatrix}^T \mathcal{L}\begin{pmatrix} y_n \\ \vdots \\ y_N \end{pmatrix}, \tag{4.1}$$

where $\mathcal{L}$ is the $(N+1-n)\times(N+1-n)$ symmetric $2n$-diagonal matrix with entries

$$\mathcal{L} = \{\mathcal{L}_{ij}\}_{i,j=0}^{N-n} = \begin{cases} \beta_{n+i}^{(n+j-i)}, & \text{if } |i-j|\le n, \\ 0, & \text{if } |i-j|>n. \end{cases} \tag{4.2}$$

If we expand in the same way the forward differences in the operator M, we get

$$M(y)_k = \Delta^n y_k - \sum_{\nu=0}^{n-1} a_k^{(\nu)}\Delta^\nu y_{k+n-\nu} =: \sum_{\nu=0}^{n}\alpha_k^{(\nu)} y_{k+\nu}.$$

Again, the sequences $\alpha^{(\nu)}$ may be computed explicitly, but we do not need these explicit expressions. Define the $(N+1)\times(N+1-n)$ matrix $\mathcal{M}$ as follows:

$$\mathcal{M} = \{M_{i,j}\} = \begin{cases} 0, & \text{if } j>i \text{ or } i-j>n, \\ \alpha_i^{(n-i+j)}, & \text{if } 0\le i-j\le n, \end{cases} \qquad i=0,\dots,N, \quad j=0,\dots,N-n.$$

Then, for any y satisfying (3.2), we have $M(y)_k = e_k^T \mathcal{M} y$, and the transpose matrix $\mathcal{M}^T$ corresponds to the adjoint operator M^*, i.e., for $y = \{y_k\}_{k=0}^N$, $M^*(y)_{k+n} = e_k^T \mathcal{M}^T y$. By a direct computation, one may verify that for any y satisfying (3.2), the relations (4.1) can be written in the matrix form

$$L(y) = \mathcal{L} y = \mathcal{M}^T \operatorname{diag}\{c_0, \dots, c_N\} \mathcal{M} y$$

and

$$\mathcal{F}(y) = y^T \mathcal{L} y = y \mathcal{M}^T \operatorname{diag}\{c_0, \dots, c_N\} \mathcal{M} y.$$

Consequently, we have the following matrix characterization of disconjugacy of (1.4). For $n = 1$, this statement may be found in [9, Section 14].

THEOREM 2. *Equation (1.4) is disconjugate on J if and only if the $(N+1-n) \times (N+1-n)$ matrix $\mathcal{L}$ given by* (4.2) *is positive definite.*

(ii) Various aspects of factorization of n^{th}-order difference operators

$$K(y) = \sum_{\nu=0}^{n} b_k^{(\nu)} y_{k+\nu}$$

are investigated in [8, Chapter IX]. It is shown there that if the equation $K(y) = 0$ is $(n-l, l)$ disconjugate, $l \in \{1, \dots, n-1\}$, (for the precise definition, see [8]) and

$$(-1)^n b_k^{(0)} > 0, \tag{4.3}$$

then there exist $(n-l)^{\text{th}}$- and l^{th}-order difference operators L_1, L_2 such that

$$K(y) = L_1(L_2(y)).$$

In contrast to these results proved under (4.3), here we impose no sign restriction on the coefficients $r_k^{(n)}$, we require only $r_k^{(n)} \neq 0$.

(iii) The fact that we do not assume any sign restriction on the sequence $r^{(n)}$ also means that the operator L and the equations $L(y) = 0$ cannot be treated in the scope of the theory of generalized linear differential equations established by Reid [10], where the *positivity* of the leading coefficient $r^{(n)}$ is required. For more details concerning this observation, see [11].

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